What is complex analysis? The main object of study is an analytic function $f:\mathbb{C}\to\mathbb{C}$. As a set, $\mathbb{C} = \mathbb{R}^2 = \{(x,y): x,y\in\mathbb{R}\}$, so you may naively think that the theory is similar to real analysis. Surprisingly, the requirement of analycity, namely, that the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists and is finite on an open set, produces results that have no counterpart in the real case. The difference is that the numbers in the expression are complex.

An example of a theorem we will prove: Louiville's Theorem

Every bounded analytic function on $\mathbb{C}$ is constant. //

Chapter 1: Complex Numbers

In this chapter, we set the stage for doing complex analysis.

Main Topics:

(1) Construct the field of complex numbers

(2) Algebraic and geometric properties

(3) Basic topological ideas of $\mathbb{C}$

Let $\mathbb{R}$ be the field of real numbers. The equation

$$x^2+1=0 \quad (*)$$

has no real solutions. We seek a field $\mathbb{C}$ containing $\mathbb{R}$ that extends the operations $+,\cdot$ of real numbers <u>and</u> contains the roots of all the polynomials. Surprisingly, the construction amounts to defining a symbol i satisfying $(*)$ and then considering all sums of the form

$$x + iy \qquad , x,y \in \mathbb{R}.$$

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Construction of the Field of Complex Numbers

Definition (The Complex Numbers) A complex number is simply an ordered pair $z=(x,y)$ of real numbers. Thus, the set of all complex numbers is given by

$$\mathbb{C} \overset{\text{def}}{=} \mathbb{R}\times\mathbb{R} = \{(x,y) : x,y \in \mathbb{R}\}.$$

If $z=(x,y)$ is a complex number, then we write

$$\operatorname{Re} z = x \quad \text{and} \quad \operatorname{Im} z = y$$

for the real and imaginary parts of z, respectively. If $\operatorname{Re} z = 0$ and $\operatorname{Im} z \neq 0$, we say that z is purely imaginary.

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Definition (Binary operations on $\mathbb{C}$) Let $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$ be complex numbers. Then their sum is

$$z_1+z_2 = (x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2),$$

and their product is

$$z_1 z_2 = (x_1,y_1)(x_2,y_2) = (x_1x_2-y_1y_2, x_1y_2+y_1x_2).$$

//

Proposition There exists a subset of $\mathbb{C}$ that is algebraically indistinguishable from $\mathbb{R}$.

Proof. Consider $A=\{(x,0) : x\in\mathbb{R}\}\subseteq\mathbb{C}$. There is a bijection

$$f:\mathbb{R}\to A, \quad x\longmapsto(x,0).$$

Moreover,

$$f(x+y) = (x+y,0) = (x,0)+(y,0) = f(x)+f(y)$$

$$f(xy) = (xy,0) = (x,0)(y,0) = f(x)f(y)$$

$\blacksquare$

According to the proposition the operations of complex addition and multiplication extend the operations of addition and multiplication of real numbers. We now identify each complex number $(x,0)$ with the corresponding real number x.

By abuse of notation, we write

$$x = (x,0).$$

Now, we define the imaginary unit as follows: $i \overset{\text{def}}{=} (0,1)$. Then

$$i^2 = i\,i = (0,1)(0,1) = (-1,0) = -1.$$

Moreover, for any $z=(x,y) \in \mathbb{C}$, we see that

$$\begin{aligned} z = (x,y) &= (x,0) + (0,y) \\ &= (x,0) + (0,1)(y,0) \\ &= x + iy. \end{aligned}$$

Hence, with our new notation:

$$\boxed{\mathbb{C} = \{x+iy \,;\, x,y \in \mathbb{R}\}}$$

with the convention that $i^2 = -1$. With this notation, the sum and product are written

$$z_1 + z_2 = (x_1+iy_1)+(x_2+iy_2) = (x_1+x_2) + i(y_1+y_2)$$

$$z_1 \cdot z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2+y_1x_2).$$

The product of complex numbers can be computed by multiplying the expressions as if they were polynomials in the variable i and using $i^2=-1$. //

Example

$$\begin{aligned} (1+i)(1-3i) &= 1 - 3i + i - 3i^2 \\ &= 1 - 3i + i + 3 \\ &= 4 - 2i. \end{aligned}$$

The proof that this works is an exercise. //

Proposition (Algebraic Properties of $(\mathbb{C}, +, \cdot)$)

(1) (Additive Identity)

$$0 + z = z = z + 0 \quad \forall z \in \mathbb{C}.$$

(2) (Associativity of Addition)

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3, \quad \forall z_i \in \mathbb{C}.$$

(3) (Commutativity of Addition)

$$z_1 + z_2 = z_2 + z_2, \quad \forall z_i \in \mathbb{C}$$

(4) (Additive Inverses) For all $z \in \mathbb{C}$, there exists a complex number denoted by $-z$ such that

$$z + (-z) = 0 = (-z) + z.$$

In fact, $-z \overset{\text{def}}{=} (-1)z$.

(5) (Multiplicative Identity)

$$1 \cdot z = z = z \cdot 1, \quad \forall z \in \mathbb{C}.$$

(6) (Associativity of Multiplication)

$$z_1(z_2 z_3) = (z_1 z_2) z_3, \quad \forall z_i \in \mathbb{C}$$

(7) (Commutativity of Multiplication)

$$z_1 z_2 = z_2 z_1, \quad \forall z_i \in \mathbb{C}$$

(8) (Multiplicative Inverses) For all $z \in \mathbb{C} \setminus \{0\}$, there exists a complex number denoted z^{-1} such that

$$z z^{-1} = 1 = z^{-1} z.$$

In fact, if $z = x + iy$, then $z^{-1} \stackrel{\text{def}}{=} \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$.

(9) (Distributive law)

$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3 \quad , \forall z_i \in \mathbb{C}.$$

//

Proof. Only (8). Let $z = x + iy$ be nonzero. Then

$$z \cdot z^{-1} = (x + iy)\left(\frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} \right)$$

$$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + i \left(-\frac{xy}{x^2+y^2} + \frac{xy}{x^2+y^2} \right)$$

$$= 1.$$

∎

In the language of algebra

(1) - (4) $(\mathbb{C}, +)$ is an abelian group.

(5) - (8) $(\mathbb{C} \setminus \{0\}, \cdot)$ is an abelian group.

(1) - (9) $(\mathbb{C}, +, \cdot)$ is a field.

The existence of additive and multiplicative inverses gives rise to subtraction and division of complex numbers.

Definition (subtraction / division) Let $z_1, z_2 \in \mathbb{C}$. We define subtraction and division as follows:

$$z_1 - z_2 \stackrel{\text{def}}{=} z_1 + (-z_2)$$

$$\frac{z_1}{z_2} \stackrel{\text{def}}{=} z_1 \cdot z_2^{-1} \quad , \quad z_2 \neq 0$$

//

The formula for z_2^{-1} is difficult to remember. In practice, division is computed by writing

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \cdot \frac{x_2-iy_2}{x_2-iy_2}$$

and multiply out the numerator and denominator. The proof is an exercise. //

$\boxed{\text{Proposition (Zero-Product Property)}}$ If $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

Proof. Assume $z_1 z_2 = 0$ and that $z_1 \neq 0$. We prove that $z_2 = 0$. Since $z_1 \neq 0$, z_1^{-1} exists. Hence,

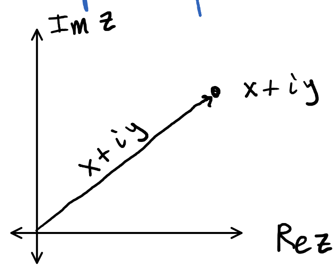
$$\begin{aligned} z_2 &= (z_1^{-1} z_1) z_2 \\ &= z_1^{-1}(z_1 z_2) \\ &= z_1^{-1} \cdot 0 \\ &= 0 \end{aligned}$$

■

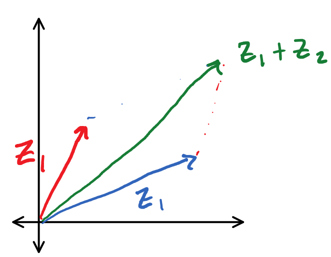
There are lots of algebraic properties in the book. Try some exercises.

Geometric Properties of Complex Numbers

As a set, $\mathbb{C} = \mathbb{R}^2$ so it is natural to visualize complex numbers as points or vectors in the complex plane



Geometrically, addition of complex numbers is just the addition of euclidean vectors



We will see a geometric interpretation of multiplication later. //

Definition (Modulus) The modulus of a complex number $z = x+iy$ is the length of the vector (x,y), namely

$$|z| \stackrel{\text{def}}{=} \sqrt{x^2+y^2}$$

//

Notice that the modulus of a real number is just the absolute value. We can immediately derive a useful inequality:

$$|z|^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 \geq (\mathrm{Re}\, z)^2 \quad (\text{or } (\mathrm{Im}\, z)^2)$$

Then (taking the sq. root)

$$\mathrm{Re}\, z \leq |\mathrm{Re}\, z| \leq |z|$$
$$\mathrm{Im}\, z \leq |\mathrm{Im}\, z| \leq |z|.$$

//

Definition (Distance) The distance between two complex numbers z_1, z_2 is

$$|z_1 - z_2|.$$

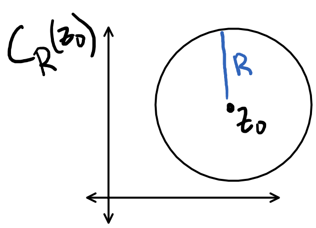
//

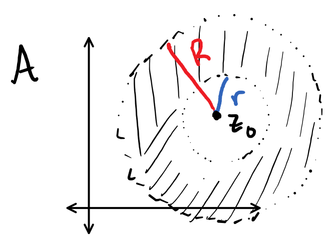
Example The modulus can be used to define various subsets of $\mathbb{C}$.

(1) The circle $C_R(z_0)$ of radius $R>0$ centered at z_0 is the set

$$C_R(z_0) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - z_0| = R\}.$$

(2) The open annulus of inner radius $r>0$ and outer radius $R>0$ centered at z_0 is the set $A = \{z \in \mathbb{C} : r < |z - z_0| < R\}$





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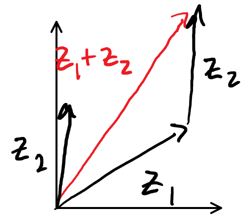
Proposition (Triangle Inequality) For all $z_1, z_2 \in \mathbb{C}$, the following inequalities hold:

(1) $|z_1 + z_1| \le |z_1| + |z_2|$

(2) $|z_1 + z_2| \ge \big|\, |z_1| - |z_2| \,\big|$



Proof. (1) Obvious fact about triangles.

(2) We need to show (a) $|z_1 + z_2| \ge |z_1| - |z_2|$ and

(b) $|z_1 + z_2| \ge -(|z_1| - |z_2|)$.

(a)
$$
\begin{aligned}
|z_1| - |z_2| &= |z_1 - z_2 + z_2| - |z_2| \\
&\overset{(1)}{\le} |z_1 + z_2| + \cancel{|-z_2|} - \cancel{|z_2|} \\
&= |z_1 + z_2|
\end{aligned}
$$

So this proves (a) when $|z_1| > |z_2|$. If $|z_1| < |z_2|$ switch the roles $|z_1 + z_1| \ge |z_2| - |z_1| = -(|z_1| - |z_2|)$. This proves (b). $\blacksquare$

$\boxed{\text{Proposition (Modulus is Multiplicative)}}$ For all $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$,

(1) $|zw| = |z||w|$

(2) $|z^n| = |z|^n$

Proof. (1) is an easy exercise.

(2) Proof of (2) is by induction. The case $n = 2$ is just (1).

Let $n \in \mathbb{N}$. Assume $|z^n| = |z|^n$. Then
$$
\begin{aligned}
|z^{n+1}| &= |z^n \cdot z| \\
&\overset{(1)}{=} |z^n| \cdot |z| \\
&\overset{\text{I.H.}}{=} |z|^n \, |z| = |z|^{n+1}.
\end{aligned}
$$
$\blacksquare$

The following lemma is a good way to demonstrate the properties of the modulus. We will use it much later to prove the Fundamental Theorem of Algebra.

$\boxed{\text{Lemma}}$ Consider the polynomial with complex coefficients

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

There exists $R>0$ such that $|z|>R$ implies

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n| R^n}.$$

(In other words, the reciprocal of a polynomial is bounded outside of a large circle $|z|=R$.)

Proof. Consider

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}, \qquad (z \neq 0).$$

Notice that $p(z) = (w + a_n) z^n$. By T.I, we get

$$|w| \le \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|}.$$

Note that, for each $0 \le k \le n-1$, the term $\frac{|a_k|}{|z|^{n-k}} \longrightarrow 0$ as $|z| \longrightarrow \infty$. This means we can choose $R>0$ such that $|z|>R$, the term $\frac{|a_k|}{|z|^{n-k}} < \frac{|a_n|}{2n}$. Then $|z|>R$ implies

$$|w| < n \frac{|a_n|}{2n} = \frac{|a_n|}{2}.$$

Then $|z|>R$ implies

$$|w+a_n| \ge \big||a_n| - |w|\big| > \left|\frac{|a_n|}{2}\right| = \frac{|a_n|}{2}.$$

Hence, $|z|>R$

$$\begin{aligned} |p(z)| &= |(w+a_n) z^n| \\ &= |w+a_n||z|^n \\ &> \frac{|a_n|}{2} R^n. \end{aligned}$$

∎